DUSO Mathematics League 2014 - 2015

Contest #1. **Calculators are not permitted on this contest.**

Part I. **ALGEBRA I** *Time Limit: 10 minutes*

The word "compute" calls for an exact answer in simplest form.

1 - 1. When 48 is subtracted from the square of a positive number, the result is 8 times the number. Compute the number.

1 - 2. At a birthday party, Joe approaches his grandfather and says, "Grandpa, we're the only two people at the party who can say that our age is one more than five times the sum of its digits!" Compute the sum of the ages of Joe and of his grandfather.

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Part II. **GEOMETRY** *Time Limit: 10 minutes*

The word "compute" calls for an exact answer in simplest form.

1 - 3. Septagon $SEPTGON$ is rotated k° counterclockwise about its center so that the figure maps onto itself. There are several possible values of k in the interval $0 < k < 360^\circ$. Compute the sum of all possible k.

1 - 4. A lattice point is a point whose coordinates are integers. How many lattice points satisfy $x^2 + y^2 < 25$?

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Part III. **ALGEBRA II / ADVANCED TOPICS** *Time Limit: 10 minutes*

The word "compute" calls for an exact answer in simplest form.

1 - 5. The value of x that solves $\sqrt{4-x} = x+1$ can be expressed in simplest form as $\dfrac{\sqrt{A}-B}{C}$ for integers A, B, and C. Compute the solution in this form.

1 - 6. Given real numbers x and y such that $x+y=5$ and $x^3+y^3=40$. Compute x^2+y^2.

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R-1. The fraction $\frac{3}{13}$ is written as a decimal. Compute the 2014th digit after the decimal point.

R-2. *Let N be the number you will receive.* Compute the positive difference between the roots of the quadratic equation $x^2 - (N+2)x + 2N = 0$.

R-3. *Let N be the number you will receive.* Working as a team, 8 mathletes can solve 15 problems in 10 minutes. At this same rate, how many mathletes does it take to solve 30 problems in N minutes?

R-4. *Let N be the number you will receive.* In base 4 arithmetic, compute the product of 21 and N. Give your answer as a base 4 number. *Note: In base 4, we count 1, 2, 3, 10, 11, 12, 13, 20, ...*

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R-5. *Let N be the number you will receive.* The pages of a book are numbered consecutively, starting with 1. If all of the digits of all of the pages are written consecutively, forming the string 12345678910111213..., what is the Nth digit in the string?

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CONTEST #1.

SOLUTIONS

1 - 1. $\boxed{\mathbf{12}}$ Solve $x^2 - 48 = 8x \to x^2 - 8x - 48 = 0 \to (x-12)(x+4) = 0$. The positive solution is **12**.

1 - 2. $\boxed{\mathbf{67}}$ Let the two-digit number TU be expressed as $10T + U$. Then, we seek to solve $10T + U = 1 + 5(T + U) \Rightarrow 5T = 4U + 1$. Note that T cannot be even, since $4U + 1$ is odd. If $T = 1$, then $U = \frac{5-1}{4} = 1$. If $T = 5$, then $U = \frac{25-1}{4} = 6$. For any other T, U is not a digit. Thus, Joe is 11 and his grandfather is 56. The sum is $11 + 56 = \mathbf{67}$.

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1 - 3. $\boxed{\mathbf{1080}}$ Since the figure has seven sides, we could rotate any integer multiple of $\frac{360}{7}$. Adding $\frac{360}{7} + \frac{2\cdot 360}{7} + \frac{3\cdot 360}{7} + \frac{4\cdot 360}{7} + \frac{5\cdot 360}{7} + \frac{6\cdot 360}{7}$ yields **1080**.

1 - 4. $\boxed{\mathbf{69}}$ Consider the first quadrant. There are four such points with $x = 1$ or $x = 2$, three lattice points with $x = 3$, and two with $x = 4$, for a total of 13. Thus, there are $4(13) = 52$ such lattice points that lie within a quadrant. There are nine on each axis, but one is the origin (and it got counted twice), so our answer is $52 + 2(9) - 1 = \mathbf{69}$.

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1 - 5. $\boxed{\dfrac{\sqrt{\mathbf{21}} - \mathbf{3}}{\mathbf{2}}}$ Square both sides to obtain $4 - x = x^2 + 2x + 1 \to x^2 + 3x - 3 = 0$, which solves to obtain $x = \frac{-3 \pm \sqrt{9 - 4\cdot 1\cdot -3}}{2}$, or $x = \frac{-3 \pm \sqrt{21}}{2}$. The desired solution is $\frac{\sqrt{\mathbf{21}} - \mathbf{3}}{\mathbf{2}}$.

1 - 6. $\boxed{\dfrac{\mathbf{41}}{\mathbf{3}}}$ Note that $x^3 + y^3 = 40 = (x+y)^3 - 3xy(x+y) = 125 - 3xy(5) = 125 - 15xy$, so $-85 = -15xy \to xy = \frac{17}{3}$. Now, consider $x^2 + y^2 = (x+y)^2 - 2xy = 25 - 2\cdot\frac{17}{3} = \frac{\mathbf{41}}{\mathbf{3}}$.

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R-1. The fraction $\frac{3}{13}$ is written as a decimal. Compute the 2014th digit after the decimal point.
R-1Sol. $\boxed{7}$ The decimal equivalent is 0.230769230769.... There are 335 occurrences of the six-digit repetend in the first 2010 digits after the decimal point. Count out four more digits; the 2014th digit is **7**.

R-2. *Let N be the number you will receive.* Compute the positive difference between the roots of the quadratic equation $x^2 - (N+2)x + 2N = 0$.
R-2Sol. $\boxed{5}$ The quadratic equation can be factored as $(x-2)(x-N) = 0$. The roots are $x = 2$ and $x = N$, whose difference is $N - 2$. Substituting, the difference desired is $7 - 2 = \mathbf{5}$.

R-3. *Let N be the number you will receive.* Working as a team, 8 mathletes can solve 15 problems in 10 minutes. At this same rate, how many mathletes does it take to solve 30 problems in N minutes?
R-3Sol. $\boxed{32}$ Since 8 mathletes can solve 15 problems in 10 minutes, 8 mathletes can solve 30 problems in 20 minutes (doubling the output by doubling the time). Now, having been passed a 5, we see that we have one-fourth the time, so we need four times as many mathletes. It would take **32** mathletes.

R-4. *Let N be the number you will receive.* In base 4 arithmetic, compute the product of 21 and N. Give your answer as a base 4 number. *Note: In base 4, we count 1, 2, 3, 10, 11, 12, 13, 20, ...*
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3-4Sol. $\boxed{1332}$ The base 4 number 21 is equivalent to $2 \cdot 4 + 1 = 9$ in base 10. Also, the number 32 is equivalent to $3 \cdot 4 + 2 = 14$ in base 10. Their base 10 product is 126, which is 1 group of 64 with 62 left over. 62 is three groups of 16 with 14 left over. 14 is three groups of 4 with 2 left over. Pass back 1332.

R-5. *Let N be the number you will receive.* The pages of a book are numbered consecutively, starting with 1. If all of the digits of all of the pages are written consecutively, forming the string 12345678910111213..., what is the Nth digit in the string?
R-5Sol. $\boxed{0}$ The first nine pages have single digits. The next ninety pages have double digits, for a subtotal of 189 digits. The remaining $N - 189 = 1332 - 189 = 1143$ digits will come from the $1143 \div 3 = 381$ pages that have triple digits. The last double-digit page number is 99, so the last of the triple digit pages in which we are interested is $99 + 381 = 480$. The Nth digit is **0**.